Deduction Rules For Propositional And First-Order Logic

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In what follows, $F_1 \ldots F_n$ are WFFs. Above the horizontal bar are the conditions for the rule to be applied, below is the result.

Propositional Rules

$\wedge$-intro

$$\frac{F_1,\ F_2}{F_1 \wedge F_2}$$

$\wedge$-elim

$$\frac{F_1 \wedge F_2}{F_2}$$

$$\frac{F_1 \wedge F_2}{F_1}$$

$\vee$-intro

$$\frac{F_1}{F_1 \vee F_2}$$

$$\frac{F_2}{F_1 \vee F_2}$$

$\vee$-elim

$$\frac{\frac{F_1}{F_3},\ \frac{F_2}{F_3},\ F_1 \vee F_2}{F_3}$$

case-elim

$$\frac{\neg\ F_1,\ F_1 \vee F_2}{F_2}$$

$$\frac{\neg\ F_2,\ F_1 \vee F_2}{F_1}$$

$\Rightarrow$-intro

$$\frac{\frac{F_1, F_2, .., F_{n-1}}{F_n}}{F_1 \wedge F2 \wedge .. \wedge F_{n-1} \Rightarrow F_n}$$

mp ***(modus ponens)***

$$\frac{F_1,\ F_1 \Rightarrow F_2}{F_2}$$

$\neg$ -intro

$$\frac{F_1}{\neg\ \neg\ F_1}$$

$\neg$ -elim

$$\frac{\neg\ \neg\ F_1}{F_1}$$

$\Box$-intro

$$\frac{F_1,\ \neg\ F_1}{\Box}$$

$\Box$-elim

$$\frac{\Box}{F_1}$$

raa ***(reductio ad absurdum)***

$$\frac{\frac{\neg\ F_1}{\Box}}{F_1}$$

Here are some other rules that may be helpful.

$\wedge$-commute

$$\frac{F_1 \wedge F_2}{F_2 \wedge F_1}$$

$\vee$-commute

$$\frac{F_1 \vee F_2}{F_2 \vee F_1}$$

DeMorgan

$$\frac{F_1 \vee F_2}{\neg\,(\neg\, F_1 \wedge \neg\, F_2)}$$

$$\frac{F_1 \wedge F_2}{\neg\,(\neg\, F_1 \vee \neg\, F_2)}$$

First-Order Rules

Note that all of the propositional rules are sound when the arguments to predicates match. For example

$$\frac{\forall\, x \bullet p(x), \neg\, p(5)}{\square}\ [\forall\text{-elim, } \square\text{-intro}]$$

quantifier-flip

$$\frac{\neg\ \forall\, x \bullet F}{\exists\, x \bullet \neg\, F}$$

$$\frac{\neg\ \exists\, x \bullet F}{\forall\, x \bullet \neg\, F}$$

$\forall$-elim

$$\frac{\forall\, x \bullet F(x)}{F(t)}\ [vars(t) \notin F(x)\ ]$$

$\exists$-elim

$$\frac{\exists\, x \bullet F(x)}{F(s)}\ [s \text{ is a Skolem function}]$$

$\forall$-intro

$$\frac{F}{\forall\, x \bullet F}\ [x \text{ not free in } F]$$

$\exists$-intro

$$\frac{F(t)}{\exists\, x \bullet F(x)}\ [x \text{ not free in } F(t)]$$