Logic and Z

Bart Massey

January 26, 2016

Formulae

A "logical formula" ("well-formed formula" or WFF) is just a grouping of symbols that is syntactically legal. The symbols are generally of two types: *atomic formulae* and *connectives*.

An atomic formula is something like $x < 5$ which can be thought of as being true exactly when x is less than 5.

A connective is something like $\wedge$ that expresses a relation between formulae, for example $x < 5 \wedge x > 7$.

A formula is *satisfiable* if there is some assignment to the variables in the atomic formulae that makes the whole formula true. We call such an assignment a *satisfying assignment*. For example, the above formula is not satisfiable.

Connectives

There are several standard logical connectives in Z. You are familiar with most of them, although the notation may not be comfortable yet. Let's start with the basics.

1. Negation: $\neg f$ is true exactly when f is false. Negation has highest precedence.

2. Conjunction: $f \wedge g$ is true exactly when f is true and g is true.

3. Disjunction: $f \vee g$ is true exactly when f is true or g is true or both. Disjunction has the same precedence as conjunction. Both are fully associative.

4. Implication: $f \Rightarrow g$ is true exactly when f is false or when f is true and g is false. Thus $f \Rightarrow g$ is the same as $\neg f \vee g$. Implication lower precedence than conjunction or disjunction and is right-associative.

5. Equivalence: $f \Leftrightarrow g$ is true exactly when $f \Rightarrow g$ is true and also $g \Rightarrow f$ is true. Thus, $f \Leftrightarrow g$ is the same as $(f \wedge g) \vee (\neg f \wedge \neg g)$. Equivalence has the same precedence as implication and is fully associative.

Interpretation

A WFF is given meaning by the meaning of its atomic symbols. Once their meaning is known, the rest follows by logic. It is really easy to screw up the model by not assigning a clear meaning to its symbols.

It is common to construct a *truth table* that shows, for each possible symbol assignment, whether it makes the formula true or false. We already did some of that above.

Sets and Quantifiers

It is often useful to reason about whole sets of things at once: even infinite sets. For this, we use the universal and existential quantifiers:

1. Universal Quantifier: The upside-down A $\forall$ is read as "for all" or "for any" and is followed by a variable that ranges over a set, then a formula. It says that for any assignment of the variable to a value in the set, the formula is true. We write something like
$$\forall\, x : S \bullet p(x)$$
where p is a predicate (formula).
2. Existential Quantifier: The upside-down E $\exists$ is read as "there exists" and is followed by a variable that ranges over a set, then a formula. It says that for at least one assignment of the variable to a value in the set, the formula is true. We write something like
$$\exists\, x : S \bullet p(x)$$
where p is a predicate (formula).
3. Unique Existential: Like the existential, but the formula holds for exactly one assignment.
$$\exists_1\, x : S \bullet p(x)$$

All of these quantifiers have a more general form. They can have multiple variables and a condition. For example:
$$\forall\, x : \mathbb{N};\; y : \mathbb{N} \mid x > y \bullet y < x$$
holds (is valid, is always true).

Relationship Between Sets and Logic

We are now in a position to describe set operations logically. For example:
$$\forall\, S, T : \mathbb{P}\,\mathbb{A} \bullet e \in (S \setminus T) \Leftrightarrow e \in S \wedge e \notin T$$

Of course, set constructors will get us there too. In fact, there is a nice symmetry here: both things always work. But it is often nice to write in a style with quantifiers.

Reordering Quantifiers

One can freely rearrange nested ∀s and their associated variables without changing the meaning. Similar for ∃s. But one must not blindly flip the two around, or bad things happen.

$$\forall x : N \bullet \exists y : N \bullet y > x \, \exists y : N \bullet \forall x : N \bullet y > x$$

Constraints and Undefinedness

Z carefully takes no position on the truth value of a formula in the situation where it says nothing. See the textbook temperature example.