1. We define the factorial function by

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ n! &= n(n-1)! \ [n > 1] \end{aligned}$$

Here is a Python program that computes this, assuming $n \geq 0$.

```python
def fact(n):
    p = 1
    # { p = 1 }
    i = 1
    # { (A) }
    while i < n:
        # { (B) }
        i = i + 1
        # { (C) }
        p = p * i
        # { (D) }
    # { i >= n, i <= n, p = i! = n! }
    return p
```

Give invariants (A), (B), (C), (D) that complete the proof that this function computes factorial.

Here is the fully annotated function.

```python
def fact(n):
    p = 1
    # { p = 1 }
    i = 1
    # { p = i! }
    while i < n:
        # { i < n, p = i! }
        i = i + 1
        # { i <= n, p = (i - 1)! }
        p = p * i
        # { i <= n, p = i * (i - 1)! = i! }
    # { i >= n, i <= n, p = i! = n! }
    return p
```

Our invariants are thus

(A)	$p = i!$
(B)	$i < n, p = i!$
(C)	$i \leq n, p = (i-1)!$
(D)	$i \leq n, p = i(i-1)! = i!$

2. A *bag* is a collection of items that can contain duplicates. Consider the following bag ADT that can only accumulate items.

$BagADT[Bag, X]$

$$new : Bag$$
$$insert : Bag \times X \to Bag$$
$$size : Bag \to \mathbb{N}$$
$$count : Bag \times X \to \mathbb{N}$$

$$size(new) = 0$$

$$\forall\, x : X \bullet \quad count(new, x) = 0$$

$$\forall\, b : Bag;\ x : X \bullet \quad size(insert(b, x)) = size(b) + 1$$

$$\forall\, b : Bag;\ x : X \bullet \quad count(insert(b, x), x) = count(b, x) + 1$$

$$\forall\, b : Bag;\ x, y : X \mid x \neq y \bullet \quad count(insert(b, x), y) = count(b, y)$$

A simple implementation of this would be to just make the bag be an unordered sequence.

$BagSeq[X]$

$$BagADT[\mathrm{seq}(X), X]$$

$$new = \langle\rangle$$

$$\forall\, b : \mathrm{seq}(X);\ x : X \bullet \quad insert(b, x) = b \frown \langle x \rangle$$

$$\forall\, b : \mathrm{seq}(X) \bullet \quad size(b) = \#\, b$$

$$\forall\, b : \mathrm{seq}(X);\ x : X \bullet \quad count(b, x) = \#(b \rhd \{x\})$$

Prove that *BagSeq* obeys each of the *Bag* laws.

$$size(new) = \#(\langle\rangle) = 0$$

$$\forall\, x : X \bullet count(new, x) = \#(\langle\rangle \rhd \{x\}) = \#(\varnothing \rhd \{x\}) = 0$$

$$\begin{aligned}&\forall\, b : \mathrm{seq}(X);\ x : X \bullet\\ &\quad size(insert(b, x)) = \#(b \frown \langle x\rangle) =\\ &\quad \#(b) + \#(\langle x\rangle) = size(b) + 1\end{aligned}$$

$$\begin{aligned}&\forall\, b : \mathrm{seq}(X);\ x : X \bullet\\ &\quad count(insert(b, x), x) =\\ &\quad \#((b \frown \langle x\rangle) \rhd \{x\}) =\\ &\quad \#(b \rhd \{x\}) + \#(\langle x\rangle \rhd \{x\}) =\\ &\quad count(b, x) + 1\end{aligned}$$

$$\begin{aligned}&\forall\, b : \mathrm{seq}(X);\ x, y : X \mid x \neq y \bullet\\ &\quad count(insert(b, x), y) =\\ &\quad \#((b \frown \langle x\rangle) \rhd \{y\}) =\\ &\quad \#(b \rhd \{y\}) + \#(\langle x\rangle \rhd \{y\}) =\\ &\quad count(b, y) + 0 = count(b, y)\end{aligned}$$